AP Calculus AB Simple Studies Review

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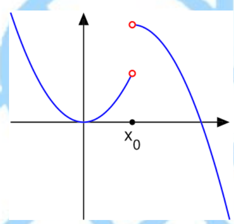
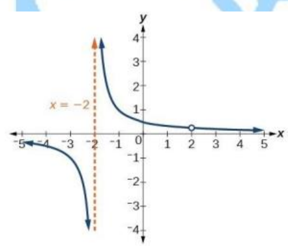
Unit 1: Limits and Continuity

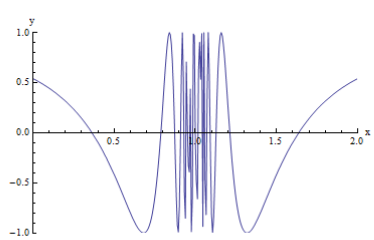
- **Limit:** The value of a function (f(x)) as it gets closer and closer to an X value.
    - Functional Notation: $\lim_{x \to a} f(x) = b$
    - Think: As the function approaches an X value, what will the Y value of the function be?
- **When do limits fail to exist?**
    - Jumps: occurs when the left and right-hand behavior at a point is different. Jumps typically occur in piecewise functions, where different functions can have different "f(x)" values at jump points.



    - Unbounded Behavior: occurs when a function approaches infinity or negative infinity (asymptote).



    - Oscillating Behavior: When a function continues to move between two points



- **Limit Properties:** used to break apart limit functions and make them easier to solve.

  Let $\lim_{x \to c} = L$ and $\lim_{x \to c} = K$

  - **Scalar Property of Limits**: $\lim_{x \to c} [\, b * f(x)] = b \lim_{x \to c} [f(x)] = bL$
  - **Sum and Difference Property of Limits**: $\lim_{x \to c} [\, f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) = L \pm K$
    - Limits that use plus signs can be represented by the *Sum Property*
    - Limits that use minus signs can be represented by the *Difference Property*
  - **Product Property of Limits**: $\lim_{x \to c} [\, f(x) * g(x)] = \lim_{x \to c} g(x) * \lim_{x \to c} f(x) = L * K$
  - **Quotient Property of Limits**: $\lim_{x \to c} [f(x)/g(x)] = \lim_{x \to c} f(x) / \lim_{x \to c} g(x) = L/K$
  - **Power Property of Limits**: $\lim_{x \to c} (f(x))^n = (\lim_{x \to c} f(x))^n = L^n$
- **Solving Indeterminate Limits Analytically:** If we plug in the C value to f(x), we may get 0/0 which is an indeterminate form, which we cannot have. In order to get a correct answer, we may have to solve the limit function analytically.
  - Factoring: Factor the numerator and denominator until we have a value that can be cancelled out at the top and bottom.
  - Conjugating: Typically used when there is a square root.
    - By multiplying the numerator and denominator by a conjugate, it will allow you to cancel out values in the top and bottom.
  - Finding a common denominator: This method is typically used when there is a fraction inside of a fraction.
    - By finding a common denominator for the inside fraction, we can then use the “keep, change, flip” method.
  - Synthetic Division:
    - 1) Set up the equation using the root of the binomial and the coefficients of the other function.
    - 2) Bring down the leading coefficient to the bottom row.
    - 3) Multiply that value by the root of the binomial and add it to the next coefficient.
    - 4) Repeat until the last value is added to the coefficient; this is your remainder.
  - Squeeze Theorem: If $h(x) < f(x) < g(x)$ for the interval, and $\lim_{x \to c} h(x) = L$ and $\lim_{x \to c} g(x) = L$, then $\lim_{x \to c} f(x) = L$.

- Trig Identities

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

- Special Trig Limits

$$\lim_{t\to 0}\frac{\sin(t)}{t} = 1$$

$$\lim_{t\to 0}\frac{1-\cos(t)}{t} = 0.$$

- Multiply by a Form of One: In order to cancel some value out, you can multiply by a form of one (# / #).

- **Left Hand Limits:** the function approaches the x value from the left side.
    - Think that the function approaches a value that is a little bit less than the value.
        - Notation: $\lim_{x\to c} f(x) = L^{-}$
        - The "-" indicates that the function is approaching that value from the left
        - Ex: $\lim_{x\to 2^-} f(x)$
            - In this example approaching two from the left hand side may mean 1.999, 1.9999, and even 1.99999 (getting closer and closer to 2 but never being 2).

- **Right Hand Limits:** the function approaches the x value from the right side.
  - Think that the function approaches a value that is a little bit more than the value.
    - Notation: $\lim_{x \to c^+} f(x) = L^+$
    - The "+" indicates that the function is approaching that value from the right
    - Ex: $\lim_{x \to 2^+} f(x)$
      - In this example 2^+ means 2.01, 2.001, 2.0001 (getting closer and closer to 2 but never being 2).
- For a limit to exist, the Right-hand limit and left-hand limit must be equal. If not, then the limit does not exist (DNE).
- **Continuity:** Must pass these conditions:
  - 1) *F(x)* is a function.
  - 2) The limit of *f(x)* exists.
  - 3) No hole and *f(c) = f(x)*
  - When a function is continuous, it can be differentiated / it has a derivative throughout the function.
- **Discontinuities**
  - Removable discontinuity: The limit resulted in a hole but *f(c) = f(x).*
    - Think: the hole can be "fixed"
  - Nonremovable Discontinuity: The function has a break, but the limit does not exist.
    - Happens at an asymptote or jump
- **Continuity Discussions**
  - List where discontinuities occur
  - Show $f(c) = \lim_{x \to c} f(x)$ or that it doesn't
  - Explain if the discontinuities are removable or nonremovable and why
- **Intermediate Value Theorem:** If *f(x)* is continuous on the closed interval [a,b], and *K* is between *f(a)* and *f(b)* then *f(c) = K* at least once.

- **Vertical Asymptotes:** The limit approaches infinity or negative infinity as it approaches the left and right side of an X value.
    - If solving a limit results in a number over 0, then there is a vertical asymptote at that X point.
    - In order to solve these limits, we must find the behavior of the function to the left and right of the asymptote.
        - First, start with the left side of the asymptote. Think: If I plug in a number a little less than the value of the asymptote, will it be positive or negative?
        - Then do the same with the right side of the asymptote. Think: If I plug in a number a little greater than the asymptote value, will it be positive or negative?
            - If both values are positive, then the answer is positive infinity.
            - If both values are negative, then the answer is negative infinity.
            - If the values are positive and negative, then the answer is DNE.
- **Horizontal Asymptotes:** describe the end behavior of the left and right ends of a graph, as the limit approaches infinity or negative infinity
    - **Horizontal Asymptote Theorem**:
        - If the highest degree in the numerator is equal to the highest degree in the denominator, then the limit of that function is equal to the coefficient of the highest degree in the numerator over that in the denominator.
        - If the highest degree of the numerator is bigger than the highest degree in the denominator OR there is a square root in function then, the answer is positive or negative infinity.
            - Think: If I plug in the *c* value (pos. or neg. infinity) to the function, will the answer be positive or negative?
                - If positive, then the answer is positive infinity
                - If negative, then the answer is negative infinity
        - If the highest degree in the numerator is smaller than the highest degree in the denominator, then the limit of that function is equal to zero.

Unit 2: Differentiation-Definition & Fundamental Properties

- **Derivative:** The slope of a tangent line made at a certain point on the function.
    - $$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
        - When trying to find the derivative of a function, plug in the function value, replacing x with x+h, and solve.
- **Tangent line:** A line that touches a function at one single point
    - The tangent line describes the slope of the function at one point
- **Differentiable:** A function is differentiable if the derivative for that function exists for every value of x.
- **Second Derivative (*F"(x)*)**: Take the derivative of a function twice.
    - This means taking the derivative of a derivative.
    - Shows us the concavity of the function - whether the graph opens up or down
- **When does a derivative not exist?**
    - Cusp: a sharp point
        - This could be at the point of an absolute value function (abs(x))
    - A point where the function is not continuous
    - Vertical Tangents
        - Ex. The left and right points at the edge of a circle

- **Derivative Shortcuts**
    - When finding the derivative of a function simply:
        - Multiply the exponent value by the X value
        - Subtract the exponent by one
- **Sum and Difference Trigonometric Identities**

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

- **Product and Quotient Rule:** Used for more complex derivatives
    - Remember: we cannot factor when taking the derivative.

Product Rule

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

- **Derivative of e^x**
    - d/dx [e^x] = e^x
    - d/dx [e^u] = e^u * *u'*
- **Derivative of Natural Log (lnx)**
    - d/dx [*lnx*] = *1/x*
    - d/dx [*lnu*] = *u'/u*
- **Derivative vs. Continuity**
    - A function is continuous if it has no breaking points - where *f(x)* exists.
    - A differentiable function must be continuous.
    - A continuous function does not need to be differentiable
        - Example: cusp, vertical tangent, or a bend.

Unit 3: Differentiation - Composite, Implicit, and Inverse Functions

- **Chain Rule**
  - 1) write out the outside function and the inside function
    - Ex. G(*f(x)*)
      - Inside function: *f(x)* ; Outside function: *G(I)*
  - 2) write down the derivative of those
    - Inside function: *f(x) = f'(x)*   O: *G(I) = G'(I)*
  - 3) Plug in the original inside equation to the derivative of the outside equation
    - G'(*f(x)*)
  - Then multiply by the derivative of the inside function
    - G'(*f(x)*) * *f'(x)*
- **Implicit Differentiation Equations:** equations with x and y values on the same side of the equal sign.
  - 1) With respect to x, take the derivative of the equation
    - For Y variables, take the derivative normally and multiply it by dy/dx
  - 2) Use algebra to get all dy/dx on to one side and x values onto the other
  - 3) Factor out the dy/dx from one side
  - 4) Divide by the remaining numbers so dy/dx is isolated
- **Second Derivative with Implicit Equations**
  - 1) Isolate dy/dx (what we did above)
  - 2) Take the second derivative of the dy/dx equation
  - 3) Plug in the equation to the dy/dx in the function
- **Derivatives of Inverse Functions**
  - **Inverse Function**: A function that switches the input and output.
  - To take an inverse, switch the x and y of the original function, and then manipulate it so you get it back to the general form (y = mx + b)

- **Inverse Function Theorem**: If *f* is differentiable at 'a' and $f(a) \neq 0$, then f^{-1} is an open interval containing $b=f(a)$ and the derivative of $f^{-1}(b)$ is:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

- **Derivative of Inverse trig functions**:

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

Unit 4: Contextual Applications of Differentiation

- **L'hopital's Rule:** If the function is differentiable and equals 0/0 or infinity/infinity, then the limit can be found by taking the derivative of the numerator function over the derivative of the denominator function.
    - ***Note that the derivatives of the numerator and denominator functions are taken *independently* of each other.
- **Velocity, Acceleration, Position**
    - The derivative of a position function gives you the velocity function.
        - Speed would be the absolute value of the derivative of a position function.
    - The derivative of a velocity function gives you the acceleration function.
- **Related Rates:** solving real life problems using derivatives
    - 1) Find the equation of the volume, according to the problem.
        - Ex.: A cube would be x^3
    - 2) Take the implicit derivative of the function with respect to time
        - Dy/dt or dx/dt
    - 3) Plug in the values given to you from the equation
    - 4) Solve.

Unit 5: Analytical Applications of Differentiation

- **Intervals of Increasing/Decreasing Functions**
  - 1) Take the derivative
  - 2) Find the critical points of the function - the points where f(x) equals zero.
  - 3) Make a Wiggle Chart
    - Make a number line with all the critical points labeled.
    - Use a test point that is between each number and on the ends.
    - Plug that point into the derivative and determine if it is positive or negative.
      - A positive value means that the original function (not the derivative) has a positive slope between those two points.
      - A negative value means that the original function has a negative slope between those two points.
- **Extrema**
  - **Extreme Value Theorem**: a continuous function must have a minimum and maximum
  - Minimums and Maximums are called Extrema or Extreme Values
  - If continuity is broken, there may not be extrema on the interval
  - Where can it occur?
    - Endpoints of an interval
    - Rounded off point on a graph
    - Derivative equals zero
    - Slope changed from positive to negative or negative to positive
    - A sharp point
  - **Absolute extrema**: The highest or lowest extreme value on an interval
  - **Relative extrema**: Every Minimum or Maximum on an interval

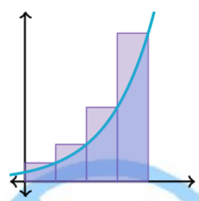
- **Concavity:** explains if a function faces up or down
    - 1) Take the second derivative of the function
    - 2) Make a second derivative chart
        - Find where all the relative extrema is - the points where the f'(x) = 0.
        - Create a number line with the relative extrema points labeled.
        - Use test points between each number and plug them into the original equation.
            - Positive means concave up, negative means concave down
    - Concave up: This means that the first derivative (slope) is increasing
    - Concave down: The slope is decreasing
    - Inflection Point: where the graph changes from concave up to down or down to up
- **Mean Value Theorem:** spot on the function where the average rate of change is the same as the slope of a tangent line.
    - The function must be continuous and differentiable on the interval

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

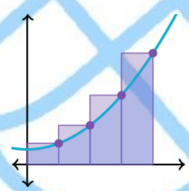
- **Rolle's Theorem:** guarantees the existence of a minimum or maximum on an interval
    - *F(x)* is continuous and differentiable on the interval [a,b]
    - *F(a) = F(b)*
    - There exists a point where the derivative is zero - there's a minimum or maximum
- **Optimization:** applied max or min
    - 1) Determine an equation for what you are trying to maximize or minimize using geometric formulas.
    - 2) Use a secondary equation that you will substitute into the primary equation
        - One variable per side
    - 3) Determine the implicit domain
    - 4) Take the derivative of both sides and set them equal to zero
    - 5) Solve

Unit 6: Integration and Accumulation of Change

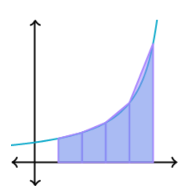
- **Riemann Sums:** area under the curves
  - Right Hand Sum
    - 1) Subtract the interval from each other and divide it by how many sub intervals that the question asks for.
    - 2) Start from the right most f(x) value and multiply it to the width.
    - 3) Repeat step 2 going from right to left
    - 4) Add the sum of all boxes together



  - Left Hand Sum
    - Do the same steps as the Right Hand sum but start from the left side and go from left to right

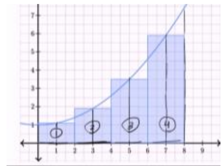


  - Trapezoidal Sum
    - 1) Subtract the interval from each other and divide it by the amount of sub intervals wanted
    - 2) Start from the left
    - 3) Width * (*f(a)*+*f(b)*) * ½
    - 4) Repeat
    - 5) Add the sum of all trapezoids together



  - Midpoint Sum

        - The same as the left and right-hand sum, but the height of each rectangle will be the midpoint of the base.



- **Integrals**
    - **Definite Integral**: the area under a specific closed interval
    - **Indefinite Integral**: represents a family of solutions
        - The answer has " + C" after it, or "plus a constant"
- **Properties of Integration**

(1) $\int_a^a f(x)\,dx = 0$

(2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(3) $\int_a^b k\,dx = k(b-a)$

where k is constant.

(4) $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

(5) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(6) If $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$

(7) If $f(x) \leq 0$ on $[a,b]$ then $\int_a^b f(x)\,dx \leq 0$

(8) If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

- **Fundamental Theorem of Calculus:** if the function is continuous on the interval and *f(x)* is the antiderivative of *f(x)* then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

- **Mean Value Theorem for Integrals**: If the function is continuous on the integral then:

$$\int_a^b f(x)dx = f(c)(b-a)$$

- **Second Fundamental Theorem of Calculus:** if the function is continuous on an open interval then:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

- **Transcendental Functions:**

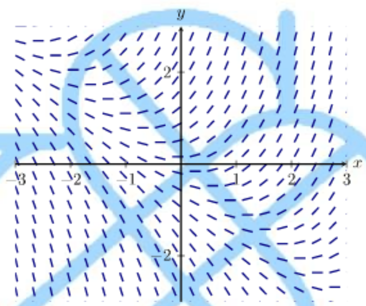
$$\int \frac{1}{x}\,dx = \ln|x| + c$$

- **U-Substitution**: reverses the chain rule
  - 1) Set a part of the function equal to *U* and differentiate it to find *du*.
  - 2) Multiply by *dx* so that *du* will be isolated
  - 3) Substitute *du* and *u* into the function
  - 4) Take the integral of the function
  - 5) Substitute the equations of *du* and *u* back in.
- **Double U-Sub**: Same as regular u substitution, except now we will need to solve for x
  - 1) Use the *u* function that we found and solve it so that we have a function of *x*
  - 2) Substitute the *du, u,* and *x* into the function
  - 3) Take the integral
  - 4) Plug the equations back into the function
- **Integrals of e^x and e^u**

$$\int e^x dx = e^x + c$$

Unit 7: Differential Equations

- Sketching Slope Fields and solution curves
  - **Slope fields**: a way to visualize the graph of a differential equation
    - You will be given a function of *dx/dy*
    - 1) Make a chart with columns x, y, and *dy/dx*
    - 2) Plug in all x and y coordinates into the *dy/dx* function
      - The resulting number will give you the *dy/dx* value or slope
    - 3) At each point (x,y), make a tangent line that has the slope that you found before
    - 4) Repeat for all points
      - It will look something like this:



  - **Solution curve:** the actual set of points that the function goes through
- **Exponential Growth and Decay**
  - *dp/dt*= KP
    - The population grows at the rate that is proportional to the population
    - *Dp/dt* is the population growth rate
    - The function for exponential growth and decay is derived from this function.
    - P is size of the population
  - The function we use for exponential growth and decay is $P(t) = P_0 e^{Kt}$
    - P_0 is the initial population
    - K is relative growth rate
    - T is time
  - Remember in order to bring an exponent down, take the natural log

- **Separable Differential Equations:** finding the equation that goes through a specific point
  - 1) Separate the equation so that all of y and dy are together and all x and dx are together.
  - 2) Integrate both sides
    - On the Y side, integrate with respect to y
    - On the X side, integrate with respect to x
  - 3) Plug in the point that was initially given to us to solve for the C value
  - 4) If your result is a ± square root:
    - Look at the initial point they gave to us
      - If the Y value is positive, we will use the positive square root
      - If the Y value is negative, we will use the negative square root

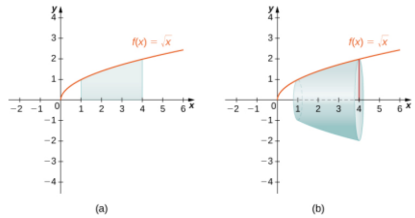
Unit 8: Application of Integration

- **Particle Motion**
  - To determine total distance traveled:
    - 1) When given the position function, take the derivative
    - 2) Find where the derivative function equals zero
      - This is where the particle changes direction
    - 3) Create a Wiggle chart
      - If the test point results in a positive number, the particle is moving to the right; if negative, then left
    - 4) Plug in two endpoints and the points where *f'(x)* is zero to the original position equation
      - This will give you the distance the particle moved in each direction
    - 5) Add together all the distances
  - If asking to find the *displacement*, make sure you consider the directions of the particles
    - If it is moving in the negative direction, subtract that distance.
- **Average Value over a closed interval**
  - **Average Value Theorem**: If the function is integrable on the interval then:

$$\frac{1}{b-a}\int_{a}^{b} f(x)dx$$

  - *b* and *a* are the endpoints of the closed interval

- **Volume by cross-sectional area - Using integrals to find volume**
  - **Disc Method**: function is rotated around x axis to form a solid



    - In order to find the area of a circle, you would use π*radius^2
      - But for this function, the height (radius) would be the function f(x), so, A=π* $f(x)^2\ dx$
      - This would only give us the area at one specific x value, so, we must take the integral of that equation in order to find the volume of the whole shape →

$$\int_a^b \pi * f(x)^2\, dx$$

  - **Washer Method**: volume between two functions rotated around the x axis
    - $$\int_a^b \pi * (g(x)^2 - f(x)^2)\, dx$$
      - *g(x)* is the function that is *greater* than *f(x)*
      - Subtracting the two will give you the volume of just the area between the two functions.

Sources

Images:

http://www.cwladis.com/math301/limitsgraphically.php

https://en.wikipedia.org/wiki/Classification_of_discontinuities

https://courses.lumenlearning.com/ivytech-collegealgebra/chapter/identify-vertical-and-horizontal-asymptotes/

https://socratic.org/questions/what-are-the-quotient-identities-for-a-trigonometric-functions

http://web.mit.edu/wwmath/calculus/limits/trig.html

http://www.mathspadilla.com/matI/Unit8-LimitsAndContinuity/limits_of_functions.html

https://calcworkshop.com/derivatives/limit-definition-of-derivative/

https://in.pinterest.com/pin/752382681468502948/

http://sites.isdschools.org/hs_math_remote_learning_resources/useruploads/math-180-ap-calculus-ab/MonMay4_StutzerHolmes_CalcAB.pdf

https://math.stackexchange.com/questions/2663733/how-to-use-the-mean-value-theorem-or-rolles-in-this-word-problem

https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-2/a/left-and-right-riemann-sums

https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-2/a/left-and-right-riemann-sums

https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-2/a/riemann-sums-review

http://pinkmonkey.com/studyguides/subjects/calc/chap7/c0707501.asp

https://www.khanacademy.org/math/in-in-grade-12-ncert/xd340c21e718214c5:definite-integrals/xd340c21e718214c5:fundamental-theorem-of-calculus-chain-rule/a/fundamental-theorem-of-calculus-review

https://calcworkshop.com/integrals/mean-value-theorem-for-integrals/

https://socratic.org/questions/how-do-you-find-the-derivative-of-sin-1-2x-1-1

Content:

https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-2-new/ab-3-6/e/second-derivatives-implicit-equations

https://apstudents.collegeboard.org/course-index-page

https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-2-new/ab-3-3/v/derivatives-of-inverse-functions

https://www.khanacademy.org/math/ap-calculus-ab/ab-integration-new/ab-6-9/a/review-applying-u-substitution

https://www.khanacademy.org/math/ap-calculus-ab/ab-applications-of-integration-new/ab-8-1/v/average-function-value-closed-interval

https://www.khanacademy.org/math/ap-calculus-ab/ab-applications-of-integration-new/ab-8-9/v/disk-method-around-x-axis

https://www.khanacademy.org/math/ap-calculus-ab/ab-applications-of-integration-new/ab-8-11/v/generalizing-the-washer-method

https://ximera.osu.edu/mooculus/calculus2/numericalMethods/digInSlopeFieldsAndEulersMethod

https://www.khanacademy.org/math/ap-calculus-ab/ab-differential-equations-new/ab-7-6/v/separable-differential-equations-introduction